POISSON SMOOTH STRUCTURES ON STRATIFIED SYMPLECTIC SPACES

HÔNG VÂN LÊ, PETR SOMBERG AND JIŘI VANŽURA

Abstract. In this note we introduce the notion of a smooth structure on a stratified space and the notion of a Poisson smooth structure on a stratified symplectic space. We show that these smooth spaces possess several important properties, e.g. the existence of smooth partitions of unity. Furthermore, under a mild condition many properties of a symplectic manifold can be extended to a symplectic stratified space provided with a smooth Poisson structure, e.g. the existence and uniqueness of a Hamiltonian flow, the isomorphism between the Brylinski-Poisson homology and the de Rham homology, the existence of a Leftschetz decomposition on a symplectic stratified space. We give many examples of stratified symplectic spaces satisfying these properties.

Contents

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1. Introduction

Many classical problems on various classes of topological spaces reduce to the quest for its appropriate functional structure. Examples of topological spaces we are interested in comprise stratified spaces equipped with an additional structure of geometrical origin. Due to the lack of canonical notion of the sheaf of smooth (or analytic) functions on such spaces one has to define a smooth structure with all derived smooth (or analytical) notions such that the obtained smooth structure satisfies good formal properties.

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In this note we continue the study of smooth structures on stratified spaces along the lines of ideas developed in [11]. One of our leading ideas is that a smooth structure on a stratified space X is defined by its behaviour on the regular strata of X, see Remark 2.12 and Remark 2.23. A large part of our note concerns with compatible smooth structures on singular spaces equipped with a stratified symplectic form. Singular spaces equipped with a stratified symplectic form are subjects of intensive study in symplectic geometry since nineties, see e.g. [19], [4], also [1], [7], where the authors considered complex manifolds $M^{2n}_{\mathbb{C}}$ with a holomorphic symplectic form ω^2, which can be turned into a symplectic form $\tilde{\omega}^2$ on differentiable manifolds $M^{2n}_{\mathbb{C}}$ of real dimension $4n$ by setting $\tilde{\omega}^2 := Re\,(\omega^2) + Im\,(\omega^2)$. Till now we have found no systematic work on smooth structures on these spaces except for symplectic orbifolds.

The structure of this note is as follows. In section 2 we introduce the notion of a smooth structure on a stratified space, see Definition 2.5. We prove that a smooth stratified space possesses several important properties, e.g. the existence of smooth partitions of unity, see Lemma 2.11, Proposition 2.17 and Corollary 2.18, which will be needed in later sections, see Remark 4.3, Remark 5.5. In section 3 we refine our notion of a smooth structure for a class of stratified spaces whose strata are symplectic manifolds, see Definition 3.2. In sections 3, 4 and 5 we show that stratified symplectic spaces (X, ω) equipped with a Poisson smooth structure which is compatible with ω possess a variety of basic properties of smooth symplectic manifolds, e.g. the existence and uniqueness of a Hamiltonian flow, the isomorphism between the Brylinski-Poisson homology and the de Rham homology, and the existence of a Leftschetz decomposition, see Theorem 3.10, Theorem 4.2, Proposition 5.2, Theorem 5.4. We also show many examples of these spaces, see Example 3.4, Example 3.7.

2. Stratified spaces and their smooth structures

In this section we introduce the notion of a stratified space following Goresky's and MacPherson's concept [9, p.36], see also [19, §1]. We introduce the notion of a smooth structure on a stratified space, see Definition 2.5. Our concept of a smooth structure on a stratified space is a natural extension of our concept of a smooth structure on a pseudomanifold with isolated conical singularities given in [11, §2], see Remark 2.12. We prove several important properties of a smooth structure on a stratified space, e.g. the existence of smooth partitions of unity and its consequences, see Lemma 2.11, Proposition 2.17, Corollary 2.18, the existence of a locally smoothly contractible, resolvable smooth structure on pseudomanifolds with edges,

see Lemma 2.20. We show that a resolvable smooth structure satisfying a mild condition is not finitely generated, see Proposition 2.24.

Definition 2.1. ([9, p.36], [19, Definition 1.1]) Let X be a Hausdorff and paracompact topological space of finite dimension and let $\mathcal{S}$ be a partially ordered set with ordering denoted by $\le$. *An $\mathcal{S}$-decomposition of X* is a locally finite collection of disjoint locally closed manifolds $S_i \subset X$ (one for each $i \in \mathcal{S}$) called *strata* such that
1) $X = \cup_{i\in\mathcal{S}} S_i$;
2) $S_i \cap \bar{S}_j \neq \emptyset \Longleftrightarrow S_i \subset \bar{S}_j \Longleftrightarrow i \le j$.
We define the depth of a stratum S as follows

$$depth_X\, S := \sup\{n|\ \text{there exist strata } S = S_0 < S_1 < \cdots < S_n\}.$$

We define the depth of X to be the number $depth\, X := \sup_{i\in\mathcal{S}} depth\, S_i$. The dimension of X is defined to be the maximal dimension of its strata.

Remark 2.2. Given a space L a cone cL over L is the topological space $L\times[0,\infty)/L\times\{0\}$. The image of $L\times\{0\}$ is the singular point of cone cL. Let $[z,t]$ denote the image of (z,t) in cL under the projection $\pi : L\times[0,\infty) \to cL$. Let $\rho_{cL} : cL \to [0,\infty)$ be defined by $\rho_{cL}([z,t]) := t$. We call ρ_{cL} the *defining function of the cone*. For any $\varepsilon > 0$ we denote by $cL(\varepsilon)$ the open subset $\{[z,t] \in cL \|\, t < \varepsilon\}$. If L has a $\mathcal{S}$-decomposition with depth n the cone cL has an induced decomposition with depth $(n+1)$ [19, p.379]. For the sake of simplicity we also denote by ρ_{cL} the restriction of ρ_{cL} to any subset of cL, if no misunderstanding occurs.

Definition 2.3. 1. (cf. [8], [19, Definition 1.7]) A decomposed space X is called *a stratified space* if the strata of X satisfy the following condition defined recursively. Given a point x in a stratum S there exist an open neighborhood $U(x)$ of x in X, an open ball B_x around x in S, a compact stratified space L, called the link of x, and *a stratified diffeomorphism* $\phi_x : U(x) \to B_x \times cL(1)$, see below for the notion of a stratified diffeomorphism.
2. A homeomorphism $\phi : X \to Y$ from a stratified space X to a stratified space Y is called *a stratified diffeomorphism*, if ϕ maps a stratum of X onto a stratum of Y and the restriction of ϕ to each stratum is a diffeomorphism on its image.
3. Let X be a stratified space. A stratum S is called *regular*, if S is open. Denote by X^{reg} the union of all regular strata. Then X^{reg} is an open subset of X and $X = \overline{X^{reg}}$. A point $x \in X^{reg}$ is called *a regular point*. Set $X^{sing} := X \setminus X^{reg}$. A point $x \in X^{sing}$ is called *a singular point*. In this note we always assume for simplicity that X^{reg} is connected.

Example 2.4. 1. A connected stratified space X of depth 1 is a disjoint union of a regular stratum X^{reg} and a countable number of strata S_i such

that $S_i \cap S_j = \emptyset$ if $i \neq j$, and $S_i \subset \overline{X^{reg}}$. Among important examples of stratified spaces of depth 1 are pseudomanifolds with edges, see e.g. [18]. Let us recall the definition of a pseudomanifold with edges. Suppose that M is a compact smooth manifold with boundary ∂M, and suppose that ∂M is the total space of a smooth locally trivial bundle $\pi : \partial M \to N$ over a closed smooth base N whose fiber is a closed smooth manifold. The topological space X obtained by gluing M with N with help of π (i.e. the points in each fiber $\pi^{-1}(s)$ are identified with $s \in N$) is called *a pseudomanifold with edges* corresponding to the pair (M, π). The natural surjective map $M \to X$ which is the identity on $M \setminus \partial M$ is denoted by $\bar{\pi}$. In general N need not be connected, and the connected components of N are called *edges* of X. Clearly $X = (X \setminus N) \cup N$ is a decomposed space, moreover $X \setminus N$ is an open connected stratum of X. Now we show that N satisfies the condition 1 in Definition 2.3. For $s \in N$ let L be the fiber $\pi^{-1}(s) \in \partial M$ and B be an open neighborhood of s in N such that $\pi^{-1}(B) = B \times \pi^{-1}(s)$. Let $\pi^{-1}(N)_\varepsilon$ be a collar neighborhood of the boundary component $\pi^{-1}(N) \subset \partial M$ in M provided with a trivialization $(p, t) : \pi^{-1}(N)_\varepsilon \to \pi^{-1}(N) \times [0, \varepsilon)$, where p is a smooth retraction $\pi^{-1}(N)_\varepsilon \to \pi^{-1}(N)$. Clearly $U(s) := \bar{\pi}(p^{-1} \circ \pi^{-1}(B))$ is a neighborhood of s and we define a trivialization $\phi_s : U(s) \to B \times cL(1)$ by $\phi_s(x) = (\pi \circ p(x), [t, p(x)])$. Hence X is a stratified space of depth 1. A pseudomanifold X with edges is called *a pseudomanifold with isolated conical singularities*, if its edges X_i are points s_i of X.

2. If L is a compact stratified space, then the cone cL is also a stratified space.

3. If L_1 and L_2 are stratified spaces, then $L_1 \times L_2$ is a stratified space. In particular, a product of cones $cL_1 \times cL_2 = c(L_1 \times L_2 \times [0,1])$ has the following decomposition: $c(L_1 \times L_2 \times [0,1]) = \{pt\} \cup L_1 \times (0,1) \cup L_2 \times (0,1) \cup L_1 \times L_2 \times (0,1)$.

4. We generalize the construction of a pseudomanifold with edges as follows. Let X be a stratified space of depth k. Let N be a proper submanifold of a stratum S of depth k in X. Suppose that $\pi : N \to B$ is a smooth fibration over a smooth submanifold B whose fiber L is compact. Then we claim that the space $Y := (X \setminus N) \cup_\pi B$ is a stratified space of depth $k + 1$. The natural surjective map $X \to Y$ which is the identity on $X \setminus N$ is denoted by $\bar{\pi}$. Clearly Y is a decomposed space, since the stratum $S \subset X$ is replaced by two new strata $S \setminus N$ and B, and we can check easy that $S \setminus N$ and B also satisfy the frontier condition (2) in Definition 2.1. Now we show that any point $x \in B$ possesses a neighborhood as in Definition 2.3. Let B be an open neighborhood of x in B such that B is diffeomorphic to an open ball and $\pi^{-1}(B) = B \times L \subset N$. Choose a Riemannian

metric on S. Let $T^{\perp}\pi^{-1}(B)$ be the normal bundle of $\pi^{-1}(B)$ in S. Denote by $T^{\perp(\varepsilon)}\pi^{-1}(B)$ the set of all normal vectors in $T^{\perp}\pi^{-1}(B)$ with length less than ε. Let $y \in \pi^{-1}(x)$. For sufficiently small ε the exponential map $\exp : T^{\perp(\varepsilon)}(\pi^{-1}(B)) \to S$ is an injective map. Let k be the codimension of N in S. Clearly $U_S(x) := \bar{\pi} \exp T^{\perp(\varepsilon)}(\pi^{-1}(B))$ is a neighborhood of x in $(S \setminus N) \cup B$ which is stratified diffeomorphic to $B_x \times c\tilde{L}(1)$, where L is a S^{k-1}-fibration over L. By assumption, any point $y \in S$ is stratified diffeomorphic to a product $B_y \times cL(1)'$, where $B_y \subset S$. It follows that a neighborhood $U(x)$ of x has the form $B_x \times c\tilde{L}(1) \times cL(1)' = B_x \times c(\tilde{L}(1) \times L(1)' \times [0,1])$. Thus Y is a stratified space of depth $k+1$.

Now let us introduce the notion of a smooth structure on a stratified space, which is a natural extension of our notion of a smooth structure on a pseudomanifold with isolated conical singularities (pseudomanifold w.i.c.s.) in [11].

We denote by $C^\infty(X^{reg})$ (resp. $C^\infty_0(X^{reg})$) the space of smooth functions on X^{reg} (resp. the space of smooth functions with compact support in X^{reg}). In the same way $C^\infty_0(S)$ denotes the space of smooth functions with compact support in S.

Note that any function $f \in C^\infty_0(X^{reg})$ has a unique extension to a continuous function, denoted by $j_* f$, on X by setting $j_*f(x) := 0$ if $x \in X \setminus X^{reg}$. The image $j_*(C^\infty_0(X^{reg}))$ is a sub-algebra of $C^0(X)$.

As the notion of a stratified space X is defined inductively on the depth of X, the notion of a smooth structure on X is also defined inductively on the depth on X. Since X is locally modeled as a product $B \times cL(1)$ we need first to define the notion of a smooth structure on the product $B \times cL$ inductively on the depth of L. For the sake of convenience we recall the definition of a smooth structure on a pseudomanifold w.i.c.s. introduced in [11].

Definition 2.5. [11, Definition 2.3] *A smooth structure* on a pseudomanifold w.i.c.s. M is a choice of a subalgebra $C^\infty(M)$ of the algebra $C^0(M)$ of all real-valued continuous functions on M satisfying the following three properties.

1. $C^\infty(M)$ is a germ-defined C^∞-ring, i.e. it is the C^∞-ring of all sections of a sheaf $SC^\infty(M)$ of continuous real-valued functions (for each open set $U \subset X$ there is a collection $C^\infty(U)$ of continuous real-valued functions on U such that the rule $U \mapsto C^\infty(U)$ defines the sheaf $SC^\infty(M)$, moreover, for any n if $f_1, \cdots f_n \in C^\infty(U)$ and $g \in C^\infty(\mathbb{R}^n)$, then $g(f_1, \cdots, f_n) \in C^\infty(U)$ [17, §1]).
2. $C^\infty(M)_{|M^{reg}} \subset C^\infty(M^{reg})$.
3. $j_*(C^\infty_0(M^{reg})) \subset C^\infty(M)$.

Remark 2.6. In [11] we showed that $C^\infty(M)$ is partially invertible in the following sense. If $f \in C^\infty(M)$ is nowhere vanishing, then $1/f \in C^\infty(M)$. This argument will be used frequently so we repeat it here. The partial invertibility property in Definition 2.5 is in fact a consequence of the first property. It suffices to show that locally $1/f$ is a smooth function. Since $f \neq 0$, shrinking a neighborhood U of x if necessary, we can assume that there is an open interval $(-\varepsilon, \varepsilon)$ which has no intersection with $f(U)$. Now there exists a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that

a) $\psi_{|(U)} = Id$,

b) $(-\varepsilon/2, \varepsilon/2)$ does not intersect with $\psi(\mathbb{R})$.

Clearly $G : \mathbb{R} \to \mathbb{R}$ defined by $G(x) = \psi(x)^{-1}$ is a smooth function. Note that $1/f(y) = G(f(y))$ for any $y \in U$. This completes the proof of our claim.

Now we will introduce the notion of a product smooth structure on a product $B \times cL(1)$, where B is a ball in $\mathbb{R}^n$.

Definition 2.7. (cf. [17, §3]) Assume that L is a compact manifold and B is an open ball in $\mathbb{R}^k$ and $C^\infty(cL(1))$ is a smooth structure on pseudomanifold w.i.c.s. $cL(1)$. *A product smooth structure* $C^\infty(X)$ on a stratified space $X := B \times cL(1)$ is the germ-defined C^∞-ring whose sheaf $SC^\infty(X)$ is generated by $\pi_1^*(SC^\infty(B))$ and $\pi_2^*(SC^\infty(cL(1)))$, where π_1 and π_2 is the projection from X to B and $cL(1)$ respectively.

Lemma 2.8. *A product smooth structure $C^\infty(B \times cL(1))$ on the stratified space $X = B \times cL(1) = B \times (L \times (0,1)) \cup B \times \{[L, 0]\}$ of depth 1 satisfies the following properties:*

1. $C^\infty(X)_{|S} \subset C^\infty(S)$ for each stratum $S \subset X$.

2. $j_*(C_0^\infty(X^{reg})) \subset C^\infty(X)$.

3. $C^\infty(X)$ is partially invertible in the following sense. If $f \in C^\infty(X)$ is nowhere vanishing, then $1/f \in C^\infty(X)$.

Proof. The first property for X^{reg} holds, since $C^\infty(X)$ is a germ-defined C^∞-ring. This condition also holds for $S = B \times \{[L, 0]\}$ since any smooth function on X locally is written as $G(f_1, \cdots f_n, h_1, \cdots h_k)$ where $G \in C^\infty(\mathbb{R}^{n+k})$, $f_i \in C^\infty(B)$, $h_i \in C^\infty(cL(1))$. Restricted to S functions h_i take constant value in $\mathbb{R}$, hence $G(f_1, \cdots, f_n, h_1, \cdots, h_k)_{|S}$ is a smooth function on some open subset in S.

The second property follows from the first property taking into account that $C^\infty(X)$ is a germ-defined C^∞-ring. The last property follows from Remark 2.6. This completes the proof of Lemma 2.8. □

Definition 2.9. Assume that a stratified space L of depth $k-1$ is equipped with a smooth structure. Let $X := cL(1)$. Denote by $C^\infty(X^{reg})$ the product smooth structure on $X^{reg} = L \times (0, 1)$. Then a smooth structure on X is a germ-defined C^∞-ring satisfying the following two properties

1. $C^\infty(X)_{|X^{reg}} \subset C^\infty(X^{reg})$.
2. $j_*(C_0^\infty(X^{reg})) \subset C^\infty(X)$.

Now we are able to introduce the notion of a smooth structure on a stratified space X of depth k recursively on k.

Definition 2.10. (cf. [11, Definition 2.3]) *A smooth structure* on a stratified space X of depth k is a choice of a $\mathbb{R}$-subalgebra $C^\infty(X)$ of the algebra $C^0(X)$ which is a germ-defined C^∞-ring satisfying the following properties.

1. $j_*(C_0^\infty(X^{reg})) \subset C^\infty(X)$.
2. For any $x \in X$ there is a local trivialization $\phi_x : U(x) \to B_x \times cL(1)$ which is a local diffeomorphism of stratified spaces, i.e. $C^\infty(U) = \phi_x^*(C^\infty(B \times cL(1)))$, where $C^\infty(B \times cL(1))$ is a product smooth structure.

Lemma 2.11. *Any smooth structure on a stratified space X satisfies the following properties.*

1. $C^\infty(X)_{|S} \subset C^\infty(S)$ for each stratum S of X.

2. $C^\infty(X)$ is partially invertible in the following sense. If $f \in C^\infty(X)$ is nowhere vanishing, then $1/f \in C^\infty(X)$.

Proof. Since $C^\infty(X)$ is germ-defined, it suffices to prove the assertion 1 in Lemma 2.11 locally. Since ϕ_x is a local diffeomorphism, it suffices to prove the assertion 1 on the local model $B \times cL(1)$ provided with a product smooth structure. Equivalently we need to show that the inclusion $i : B \times \{[L, 0]\} \to B \times cL(1)$ is a smooth map. Repeating the argument in the proof of Lemma 2.8 we obtain the smoothness of the inclusion i easily.

The second assertion of Lemma 2.11 can be proved using the argument in Remark 2.6. This completes the proof of Lemma 2.11. □

Remark 2.12. 1. Denote by i the canonical inclusion $X^{reg} \to X$. Then $i^*(C^\infty(X))$ is the restriction of smooth functions from X to X^{reg}. Since $X = \overline{X^{reg}}$, the kernel of $i^* : C^\infty(X) \to C^0(X^{reg})$ is zero. The property 1 in Lemma 2.11 implies that $i^*(C^\infty(X))$ is a subalgebra of $C^\infty(X^{reg}) \subset C^0(X^{reg})$. Roughly speaking, we can regard $C^\infty(X)$ as a subalgebra of $C^\infty(X^{reg})$.

2. The first condition in Definition 2.10 says that $j_*(C_0^\infty(X^{reg}))$ is a subalgebra of $C^\infty(X) \subset C^0(X)$. Since $\ker j_* = 0$, we can also regard $C_0^\infty(X^{reg})$ as a subalgebra of $C^\infty(X)$. The second condition in Definition 2.10 says that locally our smooth structure is equivalent to a product smooth structure of a product of a ball and a cone.

3. In the case of pseudomanifold with isolated conical singularities the second condition in Definition 2.10 is always satisfied, since we can define a smooth structure on $cL(1)$ with a help of ϕ_x, which is a local stratified diffeomorphism by assumption.

4. Our definition of a smooth structure on a stratified space stands between the definition due to Sjammar and Lerman [19], which requires a smooth structure to satisfy only the property 1 in Lemma 2.11, and the definition due to Pflaum [21], which requires a smooth structure to be obtained by gluing smooth structures on singular charts obtained from the standard smooth structure on $\mathbb{R}^n$ using some embedding of the singular chart to a vector space $\mathbb{R}^n$, in particular Pflaum needs the notion of smooth transition functions as in the definition of a smooth manifold.

Definition 2.13. A continuous map f between smooth stratified spaces $(X, C^\infty(X))$ and $(Y, C^\infty(Y))$ is called *a smooth map*, if $f^*(C^\infty(Y)) \subset C^\infty(X)$.

The following Lemma 2.14 is an important step in our proof of the existence of a partition of unity on a stratified space provided with a smooth structure. It is a generalization of [11, Lemma 2.8].

Lemma 2.14. *Let $x_0 \in X$ and let $U_\varepsilon(x_0)$ be an ε-neighborhood of x_0. Then there exists a function $f \in C^\infty(X)$ such that*

(1) *$0 \le f \le 1$ on X;*
(2) *$f(x_0) = 1$;*
(3) *$f = 0$ outside $U_\varepsilon(x_0)$.*

Proof. By definition we have Let us fix a compact stratified space L and a stratified diffeomorphism $\phi_{x_0} : U_\varepsilon(x_0) \to B(\varepsilon) \times cL(\varepsilon)$. Let $\dot{U}_\varepsilon(x)$ denote the open set $\phi_x^{-1}[(B(\varepsilon) \setminus \{0\}) \times (L \times (0, \varepsilon))] \subset U_\varepsilon(x)$. We will construct the required function $f \in C^\infty(X)$ in several steps, using the decomposition

$$X = (X \setminus U_\varepsilon(x)) \cup \dot{U}_\varepsilon(x) \cup \phi_x^{-1}(\{0\} \times cL(\varepsilon)) \cup \phi_x^{-1}(B(\varepsilon) \times [L, 0]).$$

In the first step we define an auxiliary smooth non-negative function $\chi \in C_0^\infty((0, \varepsilon))$ satisfying the following conditions

$$\chi(a) = 0 \text{ for } a \in (0, \frac{1}{5}\varepsilon], \quad 0 < \chi(a) < 1 \text{ for } a \in (\frac{1}{5}\varepsilon, \frac{2}{5}\varepsilon),$$
$$\chi(a) = 1 \text{ for } a \in [\frac{2}{5}\varepsilon, \frac{3}{5}\varepsilon], \quad 0 < \chi(a) < 1 \text{ for } a \in (\frac{3}{5}\varepsilon, \frac{4}{5}\varepsilon),$$
$$\chi(a) = 0 \text{ for } a \in [\frac{4}{5}\varepsilon, \varepsilon).$$

In the second step we define a continuous function $\chi_X \in C^0(X)$ by setting:

$$\chi_X(y) := \chi \circ (\frac{\rho_B + \rho_{cL}}{2}) \circ \phi_x(y) \text{ for } y \in \dot{U}_\varepsilon(x), \tag{2.1}$$

$$\chi_X(y) := 0 \text{ for } y \in X^{reg} \setminus \phi_x^{-1}[(B(\varepsilon) \times cL(\varepsilon))]. \tag{2.2}$$

$$\chi_X(y) := \chi \circ (\frac{\rho_{cL}}{2}) \circ \phi_x(y) \text{ if } \phi_x(y) \in \{0\} \times cL(\varepsilon), \tag{2.3}$$

$$\chi_X(y) := \chi \circ (\frac{\rho_B}{2}) \circ \phi_x(y) \text{ if } \phi_x(y) \in (B(\varepsilon) \setminus \{0\}) \times [L, 0], \tag{2.4}$$

$$\chi_X(x) := 0. \tag{2.5}$$

We note that if $y \in \dot{U}_\varepsilon(x)$, then in a small neighborhood of y in $\dot{U}_\varepsilon(x)$, the function χ_X is the restriction of a smooth function in $C^\infty(X)$, since both functions ρ_B and ρ_{cL} are smooth functions on $(B(\varepsilon)\setminus\{0\})\times(L\times(0,\varepsilon))$ and ϕ_x is a local diffeomorphism. In the same way we check that in a neighborhood of a point y, where y is a point defined in one of the cases (2.2), (2.3), (2.4), (2.5), the function χ_X is the restriction of a smooth function on X, using the construction of χ and properties 1, 2 in Definition 2.10. Since $C^\infty(X)$ is germ-determined, $\chi_X \in C^\infty(X)$.

In the third step we define a new function $\psi \in C^0(X)$. We set

$$\psi(y) := 1 \text{ for } y \in X \setminus \phi_x^{-1}[(B(\varepsilon) \times cL(\varepsilon)],$$
$$\psi(y) := 1 \text{ for } y \in U_\varepsilon(x) \text{ and } (\rho_B + \rho_{cL}) \circ \phi_x(y) > \frac{4\varepsilon}{5},$$
$$\psi(y) := \chi_X(y) \text{ for } y \in U_\varepsilon(x) \text{ and } (\rho_B + \rho_{cL}) \circ \phi_x(y) \le \frac{4\varepsilon}{5}.$$

We will show that on a neighborhood of any point $x \in X$ the function ψ coincides with a function from $C^\infty(X)$. If $y \in X \setminus \phi_x^{-1}[B(\varepsilon) \times cL(\varepsilon)]$, or $y \in U_\varepsilon(x)$ and $(\rho_B + \rho_{cL}) \circ \phi_x(y) > \frac{4\varepsilon}{5}$, then on a neighborhood of y the function ψ coincides with the constant function $1 \in C^\infty(X)$. Otherwise, on a neighborhood of y the function ψ coincides with the function $\chi_X \in C^\infty(X)$. The condition 1 in Definition 2.10 implies that $\psi \in C^\infty(X)$.

Now let $f := 1 - \psi \in C^\infty(X)$. This function has all the required properties. □

Lemma 2.15. *For every compact subset $K \subset X$ and every neighborhood U of K there exists a function $f \in C^\infty(X)$ such that*

(1) *$f \ge 0$ on X;*
(2) *$f > 0$ on K;*
(3) *$f = 0$ outside U.*

Proof. For each point $x \in K$ we take its open neighborhood $U_\varepsilon(x)$ in such a way that $U_\varepsilon(x) \subset U$, and we take a function $f_x \in C^\infty(X)$ described in Lemma 2.14 for $U_\varepsilon(x)$. Next, we take an open neighborhood $W_\varepsilon(x) \subset U_\varepsilon(x)$ of x such that $f_{x|W_\varepsilon(x)} > \frac{1}{2}$. Because K is compact, we can find a finite number of $x_1, \dots, x_r$ in K such that

$$W_{\varepsilon_1}(x_1) \cup \cdots \cup W_{\varepsilon_r}(x_r) \supset K.$$

Now it is sufficient to set $f = f_{x_1} + \cdots + f_{x_r}$. □

Lemma 2.16. *Let $\{U_i\}_{i\in I}$ be a locally finite open covering of X. Then there exists a locally finite open covering $\{V_i\}_{i\in I}$ (with the same index set) such that $\bar{V}_i \subset U_i$.*

Proof. The proof is standard. □

We are going to prove the existence of smooth partitions of unity, which is important for later applications, see Remark 4.3, Remark 5.5.

Proposition 2.17. *Let $\{U_i\}_{i\in I}$ be a locally finite open covering of X such that each U_i has a compact closure $\bar{U}_i$. Then there exists a smooth partition of unity $\{f_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$.*

Proof. Let $\{V_i\}_{i\in I}$ be the same covering as in Lemma 2.16. Let $\{W_i\}_{i\in I}$ be an open covering such that $\bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i$. According to Lemma 2.15 for every $i \in I$ there exists a function $g_i \in C^\infty(X)$ such that

(1) $g_i \geq 0$ on X;
(2) $g_i > 0$ on $\bar{V}_i$;
(3) $g_i = 0$ outside W_i.

Because $V_i \subset \operatorname{supp} g_i \subset U_i$ for every $i \in I$, the sum $g = \sum_{i\in I} g_i$ is well defined and everywhere positive. Since our algebra $C^\infty(X)$ is germ-defined, g belongs to $C^\infty(X)$, and according to the partial invertibility property in Lemma 2.11 $1/g \in C^\infty(X)$. Consequently, defining $f_i = g_i/g$, we obtain the desired partition of unity. □

Corollary 2.18. *Smooth functions on X separate points on X.*

Proof. Let $x_1, x_2 \in X$, $x_1 \neq x_2$. We take an ε-neighborhood $U_\varepsilon(x_2)$ of x_2 such that $x_1 \notin U_\varepsilon(x_2)$. Then it suffices to take a function f from Lemma 2.14 and we have $f(x_1) = 0$ and $f(x_2) = 1$. □

Example 2.19. 1. Assume that a stratified space X is embedded into a $\mathbb{R}^n$ such that each stratum S of X is a submanifold in $\mathbb{R}$. Then X has smooth structure $C^\infty(X)$ defined by $SC^\infty(X) := SC^\infty(\mathbb{R}^n)_{|X}$ satisfying our conditions in Definition 2.10. Clearly by construction $C^\infty(X)$ is a germ-defined C^∞-ring, moreover the condition 2 in Definition 2.10 is trivially satisfied. Let us check the first condition. Since X^{reg} is a submanifold in $\mathbb{R}^n$, any smooth function f with compact support on X^{reg} can be extended to a function $\tilde{f} \in C^\infty(\mathbb{R}^n)$ such that $\tilde{f}_{|X^{reg}} = f$, see e.g. [15]. Clearly $j_*(f) = \tilde{f}_{|X} \in C^\infty(X)$. Thus $C^\infty(X)$ is a smooth structure according to Definition 2.10.

2. Assume that G is a compact group acting on a differentiable manifold M. Denote by $C^\infty(M)^G$ the set of G-invariant smooth function on M. The orbit space M/G is known to be a stratified space, see e.g. [19, Example

1.8], [21, Satz 4.3.10]. The key ingredient is the existence of slices of G-action. Let $\pi : M \to M/G$ be the natural projection. Define by $C^\infty(M/G)$ the subalgebra in $C^0(M/G)$ of functions f such that $\pi^* f \in C^\infty(M)^G$ - the algebra of all G-invariant smooth functions on M. We claim that $C^\infty(M/G)$ is a smooth structure according to Definition 2.10. Clearly $C^\infty(M/G)$ is a germ-defined C^∞-ring. Now we show that $C^\infty(M/G)$ satisfies the first condition in Definition 2.10. Assume that $f \in C_0^\infty((M/G)^{reg})$ has compact support. Then $\pi^*(f) \in C^\infty(\pi^{-1}((M/G)^{reg}))$ has also compact support in the open set $\pi^{-1}((M/G)^{reg}) \subset M$. Clearly $j_*(\pi^*(f)) \in C^\infty(M)^G$. This shows that $j_*(f) \in C^\infty(M/G)$. It remains to show that $C^\infty(M/G)$ also satisfies the second condition in Definition 2.10, i.e. $C^\infty(M/G)$ locally is equivalent to a product smooth structure. This has been proved in [21, Satz 4.3.10]. In fact, it has been proved by Mather, Bierstone and Schwarz that there is an embedding $i : M/G \to \mathbb{R}^n$ such that $C^\infty(M/G) = i^*(C^\infty(\mathbb{R}^n))$, see [19] and [21] for discussion. Repeating the arguments in Example 2.19 we also conclude that $C^\infty(M/G)$ is a smooth structure according to Definition 2.10.

3. Assume that we have a continuous surjective map $M \stackrel{\pi}{\to} X$ from a smooth manifold M with corner to a stratified space X such that for each stratum $S_i \subset X$ the triple $(\pi^{-1}(S_i), \pi_i, S_i)$ is a differentiable fibration, moreover for each $x \in X^{reg}$ the preimage $\pi^{-1}(x)$ consists of a single point. The $\mathbb{R}$-subalgebra $C^\infty(X) := \{f \in C^0(X) | \, \pi^* f \in C^\infty(M)\}$ is called *a resolvable smooth structure*. We show that a resolvable smooth structure satisfies the conditions in Definition 2.10. Clearly $C^\infty(X)$ is a germ-defined C^∞-ring, since $C^\infty(M)$ satisfies these property. Since $\pi_{|\pi^{-1}(X^{reg})}$ is a diffeomorphism, the second property in Definition 2.10 follows. Finally the existence of a local smooth trivialization ϕ_x for each $x \in X$ is a consequence of the existence of a differentiable fibration $(\pi^{-1}(S_i), \pi_i, S_i)$, see also Example 2.4.4. The space M is called *a resolution* of X.

We say that $C^\infty(M)$ is *locally smoothly contractible*, if for any $x \in M$ there exists an open neighborhood $U(x) \ni x$ together with a smooth homotopy $\sigma : U(x) \times [0, 1] \to U(x)$ joining the identity map with the constant map $U(x) \mapsto x$ [17, §5]. Let $C^\infty(U(x) \times [0, 1])$ be the product smooth structure generated by $C^\infty(U(x))$ and $C^\infty([0, 1])$ [17, §3], see also Definition 2.7.

Lemma 2.20. *Every pseudomanifold X with edges has a resolvable smooth structure, which is locally smoothly contractible.*

Proof. Let X be obtained from a compact smooth manifold M with boundary ∂M, and $\bar\pi : M \to X$ - the corresponding surjective map as in Example 2.4. Example 2.19.3 shows that X has a resolvable smooth structure $C^\infty(X) := \{f \in C^0(X) | \, \bar\pi^* f \in C^\infty(M)\}$. We will show that $C^\infty(X)$

is locally smoothly contractible. Let S_i be a singular stratum of X, and $\bar{\pi}^{-1}(S_i) = \partial M_i \subset \partial M$. Let $V(\partial M_i)$ be a collar open neighborhood of ∂M_i in M. Then $U(S_i) := \bar{\pi}(V(\partial M_i))$ is an open neighborhood of S_i in X. Let us consider the following commutative diagram

$$\begin{tikzcd} I \times V(\partial M_i) \arrow[r, "\tilde{F}"] \arrow[d, "(Id\times\bar{\pi})"] & V(\partial M_i) \arrow[d, "\bar{\pi}"] \\ I \times U(S_i) \arrow[r, "F"] & U(S_i) \end{tikzcd}$$

where $\tilde{F}$ is a smooth retraction from $V(\partial M_i)$ to ∂M_i, constructed using the fibration $[0,1) \to V(\partial M_i) \to \partial M_i$. We set

$$F(t,x) := \bar{\pi}(\tilde{F}(t, \bar{\pi}^{-1}(x))).$$

Since $\tilde{F}_{|\partial M_i} = Id$, the map F is well-defined. Clearly F is a smooth homotopy, since $\tilde{F}$ is a smooth homotopy. This proves Proposition 2.20. □

Remark 2.21. A smooth structure on a stratified space X is called *Euclidean*, if for any point x there is a neighboorhood U of x together with a smooth embedding $\phi_x : U \to \mathbb{R}^n$ such that the image of $U(x)$ is a cone with vertex $\phi(x) = 0 \in \mathbb{R}^n$. It is easy to see that an Euclidean smooth structure is locally smoothly contractible, cf. Remark 2.16.2 in [11]. The main reason we cannot prove the locally smooth contractibility of a smooth structure on a cone cL is that we do now whether the map $cL \times [0,1] \to cL$, $([x,t],\tau) \mapsto [x, t\cdot\tau]$, is smooth at $([L,0],\tau)$.

Next we introduce the notion of the cotangent bundle of a stratified space X, which is identical with the notion we introduced in [11] and similar to the notions introduced in [19], [21, B.1]. Note that the germs of smooth functions $C^\infty_x(X)$ is a local $\mathbb{R}$-algebra with the unique maximal ideal $\mathfrak{m}_x$ consisting of functions vanishing at x. Set $T^*_x(X) := \mathfrak{m}_x/\mathfrak{m}_x^2$. Since the following exact sequence

$$0 \to \mathfrak{m}_x \to C^\infty_x \stackrel{j}{\to} \mathbb{R} \to 0 \tag{2.6}$$

split, where j is the evaluation map, $j(f_x) = f_x(x)$ for any $f_x \in C^\infty_x$, the space T^*_xX can be identified with the space of Kähler differentials of $C^\infty_x(X)$. The Kähler derivation $d : C^\infty_x(X) \to T^*_xX$ is defined as follows:

$$d(f_x) = (f_x - j^{-1}(f_x(x)) + \mathfrak{m}_x^2, \tag{2.7}$$

where $j^{-1} : \mathbb{R} \to C^\infty_x$ is the left inverse of j, see e.g. [12, Chapter 10], or [21, Proposition B.1.2]. We call T^*_xX *the cotangent space* of X at x. Its dual space $T^Z_xX := Hom(T^*_xX, \mathbb{R})$ is called *the Zariski tangent space* of X

at x. The union $T^*X := \cup_{x\in X}T^*_xX$ is called *the cotangent bundle* of X. The union $\cup_{x\in X}T^Z_xX$ is called *the Zariski tangent bundle* of X.

Let us denote by $\Omega^1_x(X)$ the $C^\infty_x(X)$-module $C^\infty_x(X)\otimes_{\mathbb{R}}\mathfrak{m}_x/\mathfrak{m}_x^2$. We called $\Omega^1_x(X)$ *the germs of 1-forms at* x. Set $\Omega^k_x(X) := C^\infty_x(X)\otimes_{\mathbb{R}}\Lambda^k(\mathfrak{m}_x/\mathfrak{m}_x^2)$. Then $\oplus_k\Omega^k_x(X)$ is an exterior algebra with the following wedge product

$$(f\otimes_{\mathbb{R}}dg_1\wedge\cdots\wedge dg_k)\wedge(f'\otimes_{\mathbb{R}}dg_{k+1}\wedge\cdots\wedge dg_l) = (f\cdot f')\otimes_{\mathbb{R}}dg_1\wedge\cdots\wedge dg_l, \tag{2.8}$$

where $f, f' \in C^\infty_x$ and $dg_i \in T^*_xM$.

Note that the Kähler derivation $d : C^\infty_x(X) := \Omega^0_x(X) \to \Omega^1_x(X)$ extends to the unique derivation $d : \Omega^k_x(X) \to \Omega^{k+1}_x(X)$ satisfying the Leibniz property. Namely we set

$$d(f\otimes 1) = 1\otimes df,$$
$$d(f\otimes\alpha\wedge g\otimes\beta) = d(f\otimes\alpha)\wedge g\otimes\beta + (-1)^{deg\,\alpha}f\otimes\alpha\wedge d(g\otimes\beta).$$

Definition 2.22. (cf. [17, §2]) A section $\alpha : X \to \Lambda^kT^*(X)$ is called *a smooth differential k-form*, if for each $x\in X$ there exists $U(x)\subset X$ such that $\alpha(x)$ can be represented as $\sum_{i_0i_1\cdots i_k} f_{i_0}df_{i_1}\wedge\cdots\wedge df_{i_k}$ for some $f_{i_0},\cdots,f_{i_k}\in C^\infty(X)$.

Denote by $\Omega(X) = \oplus_k\Omega^k(X)$ the space of all smooth differential forms on X. We identify the germ at x of a k-form $\sum_{i_0i_1\cdots i_k} f_{i_0}df_{i_1}\wedge\cdots\wedge df_{i_k}$ with element $\sum_{i_0i_1\cdots i_k} f_{i_0}\otimes df_{i_1}\wedge\cdots\wedge df_{i_k}\in\Omega^k_x(X)$. Clearly the Kähler derivation d extends to a map also denoted by d mapping $\Omega(X)$ to $\Omega(X)$.

Remark 2.23. Let $i^*(\Omega(X))$ be the restriction of $\Omega(X)$ to X^{reg}. By Remark 2.12 the kernel $i^* : \Omega(X)\to\Omega(X^{reg})$ is zero. Roughly speaking, we can regard $\Omega(X)$ as a subspace in $\Omega(X^{reg})$.

A C^∞-ring $C^\infty(X)$ is called *finitely generated*, if there are finite elements $f_1,\cdots,f_k\in C^\infty(X)$ such that any $h\in C^\infty(X)$ can be written as $h = G(f_1,\cdots,f_k)$, where $G\in C^\infty(\mathbb{R}^k)$. Using the notion of the cotangent space of a smooth stratified space X we will prove the following

Proposition 2.24. *A resolvable smooth structure on a stratified space X obtained from a smooth manifold M with corner is not finitely generated as a C^∞-ring, if there exists $x\in X$ such that* $\dim\pi^{-1}(x)\geq 1$, *where $\pi : M\to X$ is the associated projection.*

Proof. Assume the opposite i.e. $C^\infty(X)$ is generated by $g_1,\cdots,g_n\in C^\infty(X)$. Then $G = (g_1,\cdots,g_n)$ defines a smooth embedding $X\to\mathbb{R}^n$, so $C^\infty(X) = C^\infty(\mathbb{R}^n)/I$, where I is an ideal of $C^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ vanishing on $G(X)$ [15, p. 21, Proposition 1.5]. In particular, the cotangent space T^*_xX is a finite dimensional linear space for all $x\in X$. We will show that this assertion leads to a contradiction.

Let S be a stratum of X such that $\dim(\pi^{-1}(S)) \ge \dim S + 1$. Let $x \in S$ and $U(x)$ a small open neighborhood of x in X. Let $f \in C^\infty(U(x))$, equivalently $\pi^*(f) \in C^\infty(\pi^{-1}(U(x))$. Let $\chi : \pi^{-1}(U(x)) \to \mathbb{R}^p_+ \times \mathbb{R}^{n-p} \subset \mathbb{R}^n$ be a coordinate map on $\pi^{-1}(U(x)) \subset M$. By definition $(\chi^{-1})^*\pi^*(f) \in C^\infty(\tilde U)$ for some open set $\tilde U \subset \mathbb{R}^n$ containing $\chi(\pi^{-1}(U(x)))$. Denote by $\tilde S$ the preimage $\chi \circ \pi^{-1}(S \cap U(x))$, which is a submanifold of $\tilde U(x)$. Let us denote the restriction of $\pi \circ \chi^{-1}$ to $\tilde S$ by $\tilde\pi$. Then the triple $(\tilde S, \tilde \pi, S\cap U)$ is a smooth fibration, whose fiber $\tilde\pi^{-1}(y)$ is a smooth manifold of dimension at least 1. We note that $(\chi^{-1})^*\pi^*(f)$ belongs to the subalgebra $C^\infty(\tilde U, \tilde S, \tilde \pi)$ consisting of smooth functions on $\tilde U$ which are constant along fiber $\tilde\pi^{-1}(y)$ for all $y \in S \cap U(x)$. Now we are going to describe the space $\mathfrak{m}_s$ for $s \in S$.

Shrinking $U(x)$ we can assume that $\tilde S = \tilde U \cap \mathbb{R}^k$ and $\tilde \pi : \tilde S \to S \cap U$ is the restriction of a linear projection $\bar\pi : \mathbb{R}^k \to \mathbb{R}^l$, where $l = \dim S$, $k = \dim \tilde S$, and $S \cap U = \mathbb{R}^l \cap U$. Here we assume that U is an open set in $\mathbb{R}^n$. Let $\mathbb{R}^{n-k}$ with coordinate $\tilde x = (\tilde x^1, \cdots, \tilde x^{n-k})$ be a complement to $\mathbb{R}^k$ in $\mathbb{R}^n \supset \tilde U$, and let $\mathbb{R}^{k-l} \subset \mathbb{R}^k$ with coordinate $\tilde y = (\tilde y^1, \cdots, \tilde y^{k-l})$ be the set $\bar\pi^{-1}(0)$. We also equip the subspace $\mathbb{R}^l$ with coordinate $\tilde z = (\tilde z^1, \cdots, \tilde z^l)$. The condition $\dim \pi^{-1}(x) \ge 1$ in Proposition 2.20 is equivalent to the equality $k - l \ge 1$, in other words, y is an essential variable. Furthermore, a point $\tilde s \in \tilde S \subset \mathbb{R}^n$ has (local) coordinates with $\tilde x = 0$.

Lemma 2.25. *A function $g \in C^\infty(\tilde U)$ belongs to $C^\infty(\tilde U, \tilde S, \tilde\pi)$ if and only if g has the form*

$$g(\tilde x^1, \cdots, \tilde x^{n-k}, \tilde y, \tilde z) = \tilde x^1 g_1(\tilde x, \tilde y, \tilde z) + \cdots + \tilde x^{n-k} g_{n-k}(\tilde x, \tilde y, \tilde z) + c(\tilde z),$$

where $g_i \in C^\infty(\tilde U)$ and $c(\tilde z)$ is a smooth function on U depending only on variable $\tilde z$.

Proof. We write for $g \in C^\infty(\tilde U, \tilde S, \tilde \pi)$

$$g(\tilde x, \tilde y, \tilde z) - g(0, \tilde y, \tilde z) = \int_0^1 \frac{dg(t\tilde x, \tilde y, \tilde z)}{dt} dt = \int_0^1 \sum_{i=1}^{n-k} \frac{\partial g(t\tilde x^1, \cdots, t\tilde x^{n-k}, \tilde y, \tilde z)}{\partial (t \tilde x^i)} \tilde x_i \, dt.$$

Setting

$$g_i = \int_0^1 \frac{\partial g(t\tilde x^1, \cdots, t\tilde x^{n-k}, \tilde y, \tilde z)}{\partial(t\tilde x^i)} \, dt$$

we get $g(\tilde x, \tilde y, \tilde z) = \sum_{i=1}^{n-k} \tilde x^i g_i(\tilde x, \tilde y, \tilde z) + g(0, \tilde y, \tilde z)$. Since $g(0, \tilde y, \tilde z)$ depends only on $\tilde z$, we obtain the “only if” part of Lemma 3.6 immediately. The “if” part is trivial. This proves Lemma 3.6. □

Now let us complete the proof of Proposition 2.24. Take a point $s \in S$ and $\tilde s \in \tilde \pi^{-1}(s)$ such that $\tilde x(\tilde s) = \tilde y(\tilde s) = \tilde z(\tilde s) = 0$. By Lemma 3.6 the maximal ideal $\mathfrak{m}_s$ is a linear space generated by functions of the form $\tilde x^i g_{i,\alpha}(\tilde x, \tilde y, \tilde z), i =$

$\overline{1, n-k}$. Let us consider the sequence $S := \{\tilde{x}^1\tilde{y}^1, \cdots, \tilde{x}^1(\tilde{y}^1)^m \in \mathfrak{m}_s\}$, $m \to \infty$. If $\dim T_s^*X = \dim \mathfrak{m}_s/\mathfrak{m}_s^2 = n$, there exists a subsequence $\tilde{x}^1(\tilde{y}^1)^{k_1}, \cdots, \tilde{x}^1(\tilde{y}^1)^{k_n}$ such that $\tilde{x}^1(\tilde{y}^1)^m$ is a linear combination of $\tilde{x}^1(\tilde{y}^1)^{k_j}$ modulo $\mathfrak{m}_s^2$ for any m, which is a contradiction. This completes the proof of Proposition 2.24. $\square$

Remark 2.26. Proposition 2.24 answers question 2 we posed in [11, §5]. We observe that there are many quotient smooth structures which are finitely generated, i.e. a quotient by a smooth group action. In this case the dimension of the fiber over singular strata (e.g. the dimension of a singular orbit) is smaller than or equal to the dimension of the generic fiber (the dimension of a generic orbit, respectively).

3. Symplectic stratified spaces and compatible Poisson smooth structures

In this section we introduce the notion of a symplectic stratified space (X, ω), see Definition 3.1. We also introduce the notion of a smooth structure compatible with ω, and the notion of a Poisson smooth structure on X, see Definition 3.2. We give (new) examples of symplectic stratified spaces with compatible (Poisson) smooth structures, see Examples 3.4, 3.7 and Lemma 3.8. We prove that the Brylinski-Poisson homology of a symplectic stratified space X with a compatible Poisson smooth structure is isomorphic to the de Rham cohomology of X, see Theorem 3.10.

Definition 3.1. (cf. [19, Definition 1.12.i]) A stratified space X is called *symplectic*, if every stratum S_i is provided with a symplectic form ω_i. The collection $\omega := \{\omega_i\}$ is called *a stratified symplectic form*, or simply *a symplectic form* if no misunderstanding can occur.

Note that on each symplectic stratum (S_i, ω_i) there is a bivector G_{ω_i} which is a section of the bundle $\Lambda^2 TS_i$ such that $G_{\omega_i}(x) = \partial y_1 \wedge \partial x_1 + \cdots + \partial y_n \wedge \partial x_n$ if $\omega_i(x) = \sum_{j=1}^n dx^j \wedge dy^j$ [2, §1.1]. If we regard ω_i as an element in $End(TS_i, T^*S_i)$ and G_{ω_i} as an element in $End(T^*S_i, TS_i)$, then G_{ω_i} is the inverse of ω_i. The bivector G_{ω_i} defines a Poisson bracket on $C^\infty(S_i)$ by setting $\{f, g\}_{\omega_i} := G_{\omega_i}(df \wedge dg)$.

Definition 3.2. (cf. [11, Remark 4.8]) Let (X, ω) be a symplectic stratified space and $C^\infty(X)$ be a smooth structure on X.

1. A smooth structure $C^\infty(X)$ is said to be *weakly symplectic*, if there is a smooth 2-form $\tilde{\omega} \in \Omega^2(X)$ such that the restriction of Ω to each stratum S_i coincides with ω_i. In this case we also say that ω is *compatible with* $C^\infty(X)$, and $C^\infty(X)$ is *compatible with* ω.

2. A smooth structure $C^\infty(X)$ is called *Poisson*, if there is a Poisson structure $\{,\}_\omega$ on $C^\infty(X)$ such that $(\{f,g\}_\omega)_{|S_i} = \{f_{|S_i}, g_{|S_i}\}_{\omega_i}$ for any stratum $S_i \subset X$.

Remark 3.3. 1. Using Remark 2.23 we observe that there exists at most one 2-form $\tilde\omega \in \Omega^2(X)$ which is compatible with a given smooth structure $C^\infty(X)$.

2. The condition 2 in Definition 3.2 is equivalent to the existence of a smooth section $\tilde G_\omega$ of $\Lambda^2 T^Z(X)$, i.e. for any $\alpha \in \Omega^2(X)$ there exists a smooth function $\tilde G_\omega(\alpha) \in C^\infty(X)$ such that for any stratum $S_i \subset X$ we have

$$\tilde G_\omega(\alpha)_{|S_i} = G_{\omega_i}(\alpha_{|S_i}). \tag{3.1}$$

Indeed, the existence a section $\tilde G_\omega$ satisfying (3.1) defines a Poisson structure on $C^\infty(X)$ by setting $\{f,g\}(x) := \tilde G_\omega(df \wedge dg)(x)$. Conversely, assume that there is a Poisson structure $\{,\}_\omega$ on $C^\infty(X)$ whose restriction to each stratum S_i coincides with the given Poisson structure on S_i. We claim the map $\tilde G_{\omega_i}$ defined by (3.1) is a linear map $\tilde G : \Omega^k(X) \to \Omega^{k-2}(X)$. Since the space of smooth differential forms is germ-defined, it suffices to show the above claim locally. Note that on some neighborhood U we can write $\alpha = \sum_i f_i dg_i \wedge dh_i$, where $f_i, g_i, h_i \in C^\infty(U)$. Since the smooth structure is Poisson, we get

$$\tilde G_\omega(\alpha) = \sum_i \tilde G_\omega(f_i dg_i \wedge dh_i) = \sum_i f_i\{g_i, h_i\} \in C^\infty(U). \tag{3.2}$$

This proves our claim.

3. We claim that any linear map $\tilde G_\omega : \Omega^2(X) \to C^\infty(X)$ defined by (3.1) extends to a linear map also denoted by $\tilde G_\omega : \Omega^k(X) \to \Omega^{k-1}(X)$. First we will define this extension locally, then we will show that our linear map does not depend on a local representation of a differential form α. Let $\alpha = f_0 df_1 \wedge \cdots \wedge f_k(x)$ on an open set U. Then we set

$$\tilde G_\omega(f_0 df_1 \wedge \cdots \wedge f_k) := \sum_{1\le i<j\le k} f_0 (-1)^{i+j+1}\{f_i, f_j\} df_1 \wedge \cdots_i \cdots_j \cdots \wedge df_k. \tag{3.3}$$

Since the RHS of (3.3) is an element of $\Omega^{k-2}(U)$, element $\tilde G_\omega(\alpha)$ belongs to $\Omega^{k-2}(U)$. Finally we note that if $U \subset X^{reg}$ then the map $\tilde G_\omega$ restricted to $\Omega^k(U)$ does not depend on a local representation of $\alpha \in \Omega^k(U)$. Using Remark 2.23 we conclude that the map $\tilde G_\omega : \Omega^k(X) \to \Omega^{k-2}(X)$ does not depend on a choice of local representative of $\alpha \in \Omega^{k-2}(X)$.

Example 3.4. We assume that a compact Lie group G acts on a symplectic manifold (M,ω) with proper moment map $J : M \to \mathfrak{g}^*$. Let $Z = J^{-1}(0)$.

The quotient space $M_0 = Z/G$ is called a symplectic reduction of M. If 0 is a singular value of J then Z is not a manifold and M_0 is called *a singular symplectic reduction*. It is known that M_0 is a stratified symplectic space [19]. Let us recall the description of M_0 by Sjamaar and Lerman. For a subgroup H of G denote by $M_{(H)}$ the set of all points whose stabilizer is conjugate to H, the stratum of M of orbit type (H).

Proposition 3.5. [19, Theorem 2.1] *Let (M, ω) be a Hamiltonian G-space with moment map $J : M \to \mathfrak{g}^*$. The intersection of the stratum $M_{(H)}$ of orbit type (H) with the zero level set Z of the moment map is a manifold, and the orbit space*

$$(M_0)_{(H)} = (M_{(H)} \cap Z)/G$$

has a natural symplectic structure $(\omega_0)_{(H)}$ whose pullback to $Z_{(H)} := M_{(H)} \cap Z$ coincides with the restriction to $Z_{(H)}$ of the symplectic form ω on M. Consequently the stratification of M by orbit types induces a decomposition of the reduced space $M_0 = Z/G$ into a disjoint union of symplectic manifolds $M_0 = \cup_{H<G}(M_0)_{(H)}$.

Since J is proper, by Theorem 5.9 in [19] the regular stratum M_0^{reg} is connected. Sjamaar and Lerman also defined a "canonical" smooth structure on M_0 as follows. Set $C^\infty(M_0)_{can} := C^\infty(M)^G/I^G$, where I^G is the ideal of G-invariant functions vanishing on Z [19, Example 1.11]. We will show that $C^\infty(M_0)_{can}$ is also a smooth structure according to our Definition 2.10. Denote by π the natural projection $Z \to Z/G$. Denote by $C^\infty(Z)$ the space of smooth functions on Z defined by the natural embedding of Z to M. Since Z is closed, $C^\infty(Z) = C^\infty(M)/I_Z$, where I_Z is the ideal of smooth functions on M vanishing on Z. We claim that the space $C^\infty(Z)^G$ of G-invariant smooth functions on Z can be identified with the space $C^\infty(M_0)_{can} = C^\infty(M)^G/I^G$. Clearly $C^\infty(M)^G/I^G$ is a subspace of G-invariant smooth functions on Z. On the other hand any smooth function f on M can be modified to a G-invariant smooth function $f_G \in C^\infty(M)$ by setting

$$f_G(x) := \int_G f(g \cdot x)\mu_g$$

for a G-invariant measure μ_g on G normalized by the condition $vol(G) = 1$. So if $g \in C^\infty(Z)^G$, then g is the restriction of a G-invariant function on M. In other words we have an injective map $C^\infty(Z)^G \to C^\infty(M)^G/I^G$. Hence follows the identity $C^\infty(Z)^G = C^\infty(M_0)_{can}$. It follows that $C^\infty(M_0)_{can}$ is the quotient of the smooth structure obtained from $C^\infty(Z)$ via the projection $\pi : Z \to M_0$. In particular $C^\infty(M_0)_{can}$ is a germ-defined C^∞-ring, since $C^\infty(Z)$ is a germ-defined C^∞-ring. Now we show that $C^\infty(M_0)_{can}$ satisfies the first and second condition in Definition 2.10. We need the following local

description of M_0 together with a local description of "canonical" smooth functions on M_0. For any linear subspace N in a symplectic vector space (V, ω) denote by N^ω the linear subspace in V which is symplectic perpendicular to N.

Theorem 3.6. [19, Theorem 5.1] *Let $x \in M_0$ and $p \in \pi^{-1}(x) \subset Z$. Let H be the stabilizer of p and $V = T_p(G \cdot p)^\omega / T_p(G \cdot p)$ be the fiber at p of the symplectic normal bundle and ω_V the symplectic form on V. Let $\Phi_V : V \to \mathfrak{h}^*$ be the moment map corresponding to the linear action of H. Let $\bar{0}$ denote the image of the origin in the reduced space $\Phi_V^{-1}(0)/H$. Then a neighborhood U_1 of x in M_0 is isomorphic to a neighborhood U_2 of $\bar{0}$ in $\Phi_V^{-1}(0)/H$. More precisely there exists a homeomorphism $\phi : U_1 \to U_2$ that induces the isomorphism $\Phi^* : C^\infty(U_2) \to C^\infty(U_1)$ of Poisson algebra.*

Furthermore Sjaamar and Lerman showed that the symplectic quotient $\Phi_V^{-1}(0)/H$ is stratified diffeomorphic to a product $B \times \Phi_W^{-1}(0)/H$, where W is the symplectic perpendicular of the subspace V_H of H-fixed vectors in V, and Φ_W is the corresponding H-momentum map on W. Then quotient $\phi_W^{-1}(0)/H$ is a cone [19, (10)-(12)]. Since the smooth structure $C^\infty(U_2)$ is induced by these mappings, it follows that $C^\infty(M_0)_{can}$ satisfies the second condition of Definition 2.10. The first condition in Definition 2.10 also holds for $C^\infty(M_0)_{can}$, since $\pi^{-1}(M_0^{reg})$ is a submanifold in M, so any function $\pi^*(f) \in C_0^\infty(\pi^{-1}(M_0^{reg}))$ extends to a smooth function $\tilde{f}$ on G. Since G is compact, we can modify $\tilde{f}$ to a G-invariant smooth function $\tilde{f}_G$ on M as above, so $\pi^*(f)$ is a restriction of a function $\tilde{f}_G \in C^\infty(M)^G$. This proves that $C^\infty(M_0)_{can}$ is also a smooth structure according to our definition.

We note that, since the $\Phi_W^{-1}(0)$ is a cone in $\mathbb{R}^{2n}$, the quotient $\Phi_W^{-1}(0)/H$ with the induced quotient smooth structure is locally smoothly contractible. Thus $C^\infty(M_0)_{can}$ is locally smoothly contractible.

The smooth structure $C^\infty(M_0)_{can}$ is known to inherit the Poisson structure from M, see [19, Proposition 3.1]. We observe that $C^\infty(M_0)_{can}$ is also weakly symplectic, since by Proposition 3.5 the pull back $\pi^*(\omega_0)$ is equal to the restriction of the symplectic form ω to Z.

Example 3.7. Let us consider the closure of a nilpotent orbit in a complex semisimple Lie algebra $\mathfrak{g}$. For $x \in \mathfrak{g}$ let $x = x_s + x_n$ be the Jordan decomposition of x, where $x_n \neq 0$ is a nilpotent element, x_s is a semisimple and $[x_s, x_n] = 0$. Denote by G the adjoint group of $\mathfrak{g}$. The adjoint orbit $G(x)$ is a fibration over $G(x_s)$ whose fiber is $\mathcal{Z}_G(x_s)$-orbit of x_n, here $\mathcal{Z}_G(x_s)$ denotes the centralizer of x_s in G. Since $G(x_s)$ is a closed orbit, a neighborhood U of a point $x \in \overline{G(x)}$ is isomorphic to a product $B \times \overline{\mathcal{Z}_G}(x_s) \cdot x_n$, where B is an open neighborhood of x_s in $G(x_s)$. It is known that the closure $\overline{\mathcal{Z}_G(x_s) \cdot x_n}$ is a finite union of $\mathcal{Z}_G(x_s)$-orbits of nilpotent elements in the

Lie subalgebra $\mathcal{Z}_{\mathfrak{g}}(x_s)$ [5, chapter 6], so the closure $\overline{G(x)}$ is a finite union of adjoint orbits in $\mathfrak{g}$ provided with the Kostant-Kirillov symplectic structure. Thus $\overline{G(x)}$ is a decomposed space, whose strata are symplectic manifolds. Moreover $\overline{G(x)}^{reg} = G(x)$ is connected.

1. Now assume that x_n is a minimal nilpotent element in $\mathcal{Z}_{\mathfrak{g}}(x_s)$. Then $\overline{G(x)}$ is a stratified symplectic space of depth 1, since $\overline{\mathcal{Z}_G(x_s) \cdot x_n} = \mathcal{Z}_G(x_s) \cdot x_n \cup \{0\}$, [5, §4.3]. The embedding $\overline{G(x)} \to \mathfrak{g}$ provides $\overline{G(x)}$ with a natural finitely generated C^∞-ring

$$C_1^\infty(\overline{G(x)}) := \{f \in C^0(\overline{G(x)}) | \, f = \tilde{f}_{|\overline{G(x)}} \text{ for some } \tilde{f} \in C^\infty(\mathfrak{g})\}.$$

Moreover $C_1^\infty(\overline{G(x)})$ satisfies the first condition of Definition 2.10, since $G(x)$ is locally closed subset in $\mathfrak{g}$. The second condition in Definition 2.10 also holds, since $\overline{G(x)}$ is a fibration over $G(x_s)$ whose fiber is the cone $\overline{\mathcal{Z}_{G(x_s)}(x_n)}$. Thus $C_1^\infty(\overline{G(x)})$ is a smooth structure according to Definition 2.10. The smooth structure $C_1^\infty(\overline{G(x)})$ is Poisson, inherited from the Poisson structure on $\mathfrak{g}$. It is also compatible with the symplectic structure on $\overline{G(x)}$, since the symplectic structure on $\overline{G(x)}$ is the restriction of the smooth 2-form $\omega_x(v, w) = \langle x, [v, w]\rangle$ on $\mathfrak{g}$. Using the argument in Remark 2.21 we can show that $C_1^\infty(\overline{G(x)})$ is locally smoothly contractible.

2. We still assume that x_n is minimal in $\mathcal{Z}_{\mathfrak{g}}(x_s)$. In [20, Lemma 2] Panyushev showed that $\overline{G(x)}$ possesses an algebraic resolution of the singularity at $(x_s, 0) \in \overline{G(x)} \subset \mathfrak{g}$. We will show that this resolution brings a resolvable smooth structure $C_2^\infty(\overline{G(x)})$ which is compatible with ω as follows. Let us recall the construction in [20]. Let h be a characteristic of x_n (i.e. $h \in \mathcal{Z}_{\mathfrak{g}}(x_s)$ is a semisimple element and $(h, x_n, y_n) \subset \mathcal{Z}_{\mathfrak{g}}(x_s)$ is an $\mathfrak{sl}_2$-triple). Let $\mathcal{Z}_{\mathfrak{g}}(x_s)(i) := \{s \in \mathcal{Z}_{\mathfrak{g}}(x_s) | [h, s] = is\}$, $\mathfrak{n}_2(x_s) := \oplus_{i \ge 2} \mathcal{Z}_{\mathfrak{g}}(x_s)(i)$, $P(x_s)$ - a parabolic subgroup with the Lie algebra $lP(x_s) := \oplus_{i\ge 0} \mathcal{Z}_{\mathfrak{g}}(x_s)(i)$ and N_- - the connected Lie subgroup of $\mathcal{Z}_G(x_s)$ with Lie algebra $lN_-(x_s) := \oplus_{i<0} \mathcal{Z}_{\mathfrak{g}}(x_s)(i)$. It is known that $\overline{P(x_s) \cdot x_n} = \mathfrak{n}_2(x_s)$ and $\mathcal{Z}_{\mathcal{Z}_{G(x_s)}}(x_n) \subset P(x_s)$. Hence there exists a natural map

$$\tau_{x_s} : \mathcal{Z}_G(x_s) *_{P(x_s)} \mathfrak{n}_2(x_s) \to \overline{\mathcal{Z}_G(x_s) \cdot x_n}, g * n \mapsto gn,$$

which is a resolution of the singularity of the cone $\overline{\mathcal{Z}_G(x_s) \cdot x_n}$. The resolution of the $\overline{G(x)}$ is obtained by considering the fibration $\mathcal{Z}_G(x_s) *_{P(x_s)} \mathfrak{n}_2(x_s) \to F \to G(x_s)$. The map τ extends to a map $\tilde{\tau} : F \to \overline{G(x)}$ as follows

$$\tilde{\tau}(x_s, y) := \tau_{x_s}(y), \text{ for } y \in \mathcal{Z}_G(x_s) *_{P(x_s)} \mathfrak{n}_2(x_s).$$

The resolvable smooth structure $C_2^\infty(\overline{G(x)})$ is defined by $\tilde{\tau}$, see Example 2.19.3 , since the minimality of x_n implies that the preimage $\tau^{-1}(\overline{\mathcal{Z}_G(x_s) \cdot x_n} \setminus$

$\mathcal{Z}_G(x_s)\cdot x_n)$ is a Lagrangian submanifold in $\mathcal{Z}_G(x_s)*_{P(x_s)}\mathfrak{n}_2(x_s) = T^*(\mathcal{Z}_G(x_s)/P(x_s))$, see [1, §2], [20]. By Lemma 2.20 $C^\infty_2(\overline{G(x)})$ is locally smoothly contractible.

3. In addition now we assume that $x_s = 0$. In this case it is shown in [20] that $\tau^*(\omega)$ is a smooth 2-form on $G *_P \mathfrak{n}_2$. It follows that $C^\infty_2(\overline{G(x)})$ is compatible with the given symplectic structure on $\overline{G(x)}$. Panyushev also showed that $\tau^*(\omega)$ is symplectic if and only if x is even. (We refer the reader to [5] and [7] for a detailed description of nilpotent orbits.) Hence $C^\infty_2(\overline{G(x)})$ is Poisson by the following Lemma 3.8.

Lemma 3.8. *Assume that X is a stratified symplectic space with isolated conical singularities and $(\tilde{X}, \tilde{\omega}, \pi : \tilde{X} \to X)$ a smooth resolution of X such that $\tilde{\omega}$ is a symplectic form on $\tilde{X}$ and $\pi^*(\omega_{|X^{reg}}) = \tilde{\omega}_{|\pi^{-1}(X^{reg})}$. If for each singular point $x \in X$ the preimage $\pi^{-1}(x)$ is a coisotropic submanifold in $\tilde{X}$, then the obtained resolvable smooth structure $C^\infty(X)$ is Poisson.*

Proof. We define a Poisson bracket on $C^\infty(X)$ by setting $\{g, f\}_\omega(x) := \{\pi^*g, \pi^*f\}_{\tilde{\omega}}(\tilde{x})$, for $\tilde{x} \in \pi^{-1}(x)$. We will show that this definition does not depend on the choice of a particular $\tilde{x}$. By definition $\{\pi^*g, \pi^*f\}_{\tilde{\omega}}(\tilde{x}) := G_{\tilde{\omega}}(d\pi^*g, d\pi^*f)(\tilde{x})$. Since π^*f and π^*g are constant along the coisotropic submanifold $\pi^{-1}(x)$, we get $G_{\tilde{\omega}}(d\pi^*g, d\pi^*f)(\tilde{x}) = 0$. This proves Lemma 3.8. ∎

Let us study the Brylinski-Poisson homology of a stratified symplectic space X equipped with a Poisson smooth structure. By Remark 2.23 the space $i^*(\Omega(X)) \cong \Omega(X)$ is a linear subspace in $\Omega(X^{reg})$. Assume that $C^\infty(M)$ is a Poisson smooth structure.

We consider the *canonical complex*

$$\to \Omega^{n+1}(X) \stackrel{\delta}{\to} \Omega^n(X) \to ...,$$

where δ is a linear operator defined as follows. Let $\alpha \in \Omega(X)$ and $\alpha = \sum_j f^j_0 df^j_1 \wedge df^j_p$ be a local representation of α as in Definition 2.22. Then we set (see [10], [2]):

$$\delta(f_0 df_1 \wedge \cdots \wedge df_n) = \sum_{i=1}^n (-1)^{i+1}\{f_0, f_i\}_\omega df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_n$$

$$+ \sum_{1\le i<j\le n} (-1)^{i+j} f_0 d\{f_i, f_j\}_\omega \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_n.$$

Lemma 3.9. *The boundary operator δ fulfills*
1. $\delta = i(G_\omega) \circ d - d \circ i(G_\omega)$. In particular, δ is well-defined.
2. $\delta^2 = 0$.

Proof. 1. The first assertion of Lemma 3.9 has been proved for any smooth Poisson manifold M by Brylinski, [2, Lemma 1.2.1]. Since we also have $\{f, g\}_\omega = G_\omega(df \wedge dg)$ on X^{reg}, using the fact that $\Omega(X^{reg}) \supset i^*(\Omega(X)) \cong \Omega(X)$, taking into account Remark 3.3 we obtain the first assertion.

2. To prove the second assertion we note that $\delta^2(\alpha)(x) = 0$ at all $x \in X^{reg}$, since δ is local operator by the first assertion. Hence $\delta^2(\alpha)(x) = 0$ for all $x \in X$. □

In general it is very difficult to compute the Brylinski-Poisson homology of a Poisson manifold M unless it is a symplectic manifold. The following theorem shows that the isomorphism between the Poisson homology and the de Rham homology on smooth stratified symplectic space X is a consequence of the compatibility of the Poisson smooth structure $C^\infty(X)$ with ω. The following Theorem 3.10 generalizes [11, Corollary 4.1] for the case of symplectic pseudomanifolds X with isolated conical singularities.

Theorem 3.10. *Suppose (X^{2n}, ω) is a stratified symplectic space equipped with a Poisson smooth structure $C^\infty(X^{2n})$, which is compatible with the symplectic form ω. Then the Brylinski-Poisson homology of the complex $(\Omega(X^{2n}), \delta)$ is isomorphic to the de Rham cohomology of X^{2n} with inverse grading : $H_k(\Omega(X^{2n}), \delta) = H^{2n-k}(\Omega(X^{2n}), d)$. If the smooth structure $C^\infty(X^{2n})$ is locally smoothly contractible, $H_k(\Omega(X^{2n}), \delta)$ is equal to the singular cohomology $H^{2n-k}(X^{2n}, \mathbb{R})$.*

Before the proof of Theorem 3.10 we recall the definition of the symplectic star operator
$$*_\omega : \Lambda^p(\mathbb{R}^{2n}) \to \Lambda^{2n-p}(\mathbb{R}^{2n})$$
which satisfies the following condition
$$\beta \wedge *_\omega \alpha = G^k(\beta, \alpha) vol,$$
where $vol = \omega^n/n!$. Now let us consider a stratified symplectic space (X^{2n}, ω) with a Poisson smooth structure $C^\infty(X^{2n})$. The operator $*_\omega : \Lambda^p T^*_x X^{reg} \to \Lambda^{2n-p} T^*_x X^{reg}$ extends to a linear operator $*_\omega : \Omega^p(X^{reg}) \to \Omega^{2n-p}(X^{reg})$. By Remark 2.12 the space $\Omega^p(X^{2n}) \cong i^*(\Omega(X^{2n})) \subset \Omega^p(X^{reg})$. In particular, we have $*_\omega(i^*(\Omega^p(X^{2n}))) \subset \Omega^{2n-p}(X^{reg})$.

Proposition 3.11. *If ω is compatible with a Poisson smooth structure $C^\infty(X^{2n})$, then $*_\omega(i^*(\Omega^k(X^{2n}))) = i^*(\Omega^{2n-k}(X^{2n}))$.*

Proof. We set $\Omega_A(X^{2n}) := \{\gamma \in \Omega(X^{2n}) |\ *_\omega i^*(\gamma) \in i^*(\Omega(X^{2n}))\}$. To prove Proposition 3.11 it suffices to show that $\Omega_A(X^{2n}) = \Omega(X^{2n})$. Note that the $C^\infty(X^{2n})$-module $\Omega^{2n}(X^{2n})$ is generated by ω^n since ω^n is smooth with respect to $C^\infty(X^{2n})$ and $C^\infty(X^{reg})$-module $\Omega^{2n}(X^{reg})$ is generated by ω^n. Furthermore, $*_\omega(i^* f) = i^*(f) i^*(\omega^n)$ for any $f \in C^\infty(X^{2n})$. This proves

$*_\omega(i^*(C^\infty(X^{2n}))) = i^*(\Omega^{2n}(X^{2n}))$. In particular $\Omega^0(X^{2n}) \subset \Omega_A(X^{2n})$, and $\Omega^{2n}(X^{2n}) \subset \Omega_A(X^{2n})$.

Lemma 3.12. *We have*

$$*_\omega(i^*(\Omega_A(X^{2n}))) = i^*(\Omega_A(X^{2n})).$$

Proof. Let $\gamma \in \Omega_A(X^{2n})$. By definition $*_\omega(i^*\gamma) = \beta \in i^*(\Omega(X^{2n}))$. Using the identity $*_\omega^2 = Id$, see [2, Lemma 2.1.2], we get $*_\omega\beta = i^*\gamma$. It follows that $\beta \in i^*(\Omega_A(X^{2n}))$. This proves $*_\omega(i^*(\Omega_A(X^{2n}))) \subset i^*(\Omega_A(X^{2n}))$. Taking into account $*_\omega^2 = Id$, this proves Lemma 3.12. □

Lemma 3.13. $\Omega_A(X^{2n})$ *has the following properties:*
1. $\Omega_A(X^{2n})$ *is a* $C^\infty(X^{2n})$*-module.*
2. $d(\Omega_A(X^{2n})) \subset \Omega_A(X^{2n})$.

Proof. 1. The first assertion of Lemma 3.13 follows from the identity $*_\omega(i^*(f(x)\cdot \phi(x))) = i^*(f(x)) \cdot *_\omega i^*(\phi(x))$ for any $f \in C^\infty(X^{2n})$, and $\phi \in \Omega(X^{2n})$ and the fact that $\Omega(X^{2n})$ is a $C^\infty(X^{2n})$-module.

2. To prove the second assertion of Lemma 3.13 it suffices to show that for any $\gamma \in \Omega_A(X^{2n})$ we have $*_\omega(i^*(d\gamma)) \in i^*(\Omega(X^{2n}))$. Using Lemma 3.12 we can write $i^*(\gamma) = *_\omega\beta$ for some $\beta \in i^*(\Omega_A(X^{2n}))$. Since $\beta \in \Omega(X^{reg})$, we can apply the identity $\delta\beta = (-1)^{deg\,\beta+1} *_\omega d*_\omega$ [2, Theorem 2.2.1], which implies

$$*_\omega i^*((d\gamma)) = *_\omega d *_\omega \beta = (-1)^{deg\,\beta+1}\delta(\beta) \in i^*(\Omega(X^{2n})),$$

since $i^* \circ \delta = \delta \circ i^*$. Hence $(d\gamma) \in \Omega_A(X^{2n})$. This proves the second assertion. □

Let us complete the proof of Proposition 3.11. Since $\Omega^1(X^{2n})$ is a $C^\infty(X^{2n})$-module, whose generators are differentials df, $f \in C^\infty(X^{2n})$, using Lemma 3.13 we obtain that $\Omega^1(X^{2n}) \subset \Omega_A(X^{2n})$. Inductively, we observe that $\Omega^k(X^{2n})$ is a $C^\infty(X^{2n})$-module whose generators are the k-forms $d\phi(x)$, where $\phi(x) \in \Omega^{k-1}(X^{2n})$. By Lemma 3.13, $\Omega^k(X^{2n}) \subset \Omega_A(X^{2n})$ if $\Omega^{k-1}(X^{2n}) \subset \Omega_A(X^{2n})$. This completes the proof of Proposition 3.11. □

Proof of Theorem 3.10. By Proposition 3.11 the equality

$$\delta\beta = (-1)^{deg\,\beta+1} *_\omega d*_\omega \tag{3.4}$$

holds also for $\Omega(X^{2n})$, since it holds for $\Omega(X^{reg})$ [2, Theorem 2.1.1]. Clearly the first assertion of Theorem 3.10 follows immediately from (3.4).

The second assertion of Theorem 3.10 follows from [17, Theorem 5.2]. This completes the proof of Theorem 3.10. □

4. The existence of Hamiltonian flows

In this section we prove the existence and uniqueness of a Hamiltonian flow associated with a smooth function H on a symplectic stratified space X equipped with a Poisson smooth structure, see Theorem 4.2. This Theorem generalizes a result by Sjamaar and Lerman in [19, §3], see also Remark 4.3.

Let (X, ω) be a stratified symplectic space and $C^\infty(X)$ a Poisson smooth structure on X.

Lemma 4.1. *For any $H \in C^\infty(X)$ the associated Hamiltonian vector field X_H defined on X by setting*

$$X_H(f) := \{H, f\}_\omega \text{ for any } f \in C^\infty(X)$$

is a smooth Zariski vector field on X. If x is a point in a stratum S, then $X_H(x) \in T_xS$.

Proof. By definition of a Poisson structure, the function $X_H(f)$ is smooth for all $f \in C^\infty(X)$. Hence X_H is a smooth Zariski vector field. This proves the first assertion of Lemma 4.1. To prove the second assertion it suffices to show that, if the restriction of a function $f \in C^\infty(X)$ to a neighborhood $U_S(x) \subset S$ of a point $x \in S$ is zero, then $X_H(f)(x) = 0$. The last identity holds, since $X_H(f)(x)$ is equal to the Poisson bracket of the restriction of H and f to S. This completes the proof. □

The following theorem generalizes a result by Sjamaar and Lerman [19, §3].

Theorem 4.2. *Given a Hamiltonian function $H \in C^\infty(X)$ and a point $x \in X$ there exists a unique smooth curve $\gamma : (-\varepsilon, \varepsilon) \to X$ such that for any $f \in C^\infty(X)$ we have*

$$\frac{d}{dt} f(\gamma(t)) = \{H, f\}.$$

The decomposition of X into strata of equal dimension can be defined by the Poisson algebra of smooth functions.

Proof. For $x \in S$ we define $\gamma(t)$ to be the Hamiltonian flow on S defined by X_H, which is by Lemma 4.1 a smooth vector field on S. This proves the existence of the required Hamiltonian flow. Now let us prove the uniqueness of the Hamiltonian flow using Sjamaar's and Lerman's argument in [19, §3]. Denote by Φ_t the Hamiltonian flow whose existence we just proved. Clearly for any $x \in X$ and a compact neighborhood $U(x)$ of $x \in X$ there exists $\varepsilon > 0$ such that $\Phi_t(x')$ is defined for all $t \le \varepsilon$ and for all $x' \in U(x)$. Let $x \in X$ and $\gamma(t)$, $t \in (-\varepsilon_1, \varepsilon_1)$ be an integral curve of X_H with $\gamma_0(0) = x$. We will show that $\Phi_t(\gamma(t)) = x_0$ for all $0 \le t \le \min(\varepsilon, \varepsilon_1)$. By Corollary

2.18.2 smooth functions on X separate points. Therefore it suffices to show that for all $t \le \min(\varepsilon, \varepsilon_1)$ and for all $f \in C^\infty(X)$ we have

$$f(\Phi_t(\gamma_t(t))) = f(x). \tag{4.1}$$

Now we compute

$$\frac{d}{dt} f(\Phi_t(\gamma(t))) = \{H, f\}_\omega(\gamma(t)) + \{f, H\}_\omega(\gamma(t)) = 0,$$

which implies (4.1). This completes the proof of the uniqueness of the Hamiltonian flow.

The last assertion of Theorem 4.2 follows from the inclusion $X_H(x) \in S$, if $x \in S$. □

Remark 4.3. In [19, §3] Sjamaar and Lerman used a slightly different method for their proof of the existence and uniqueness of a Hamiltonian flow on singular symplectic reduced space $(M^{2n}, \omega)//G$. For the existence of a Hamiltonian flow they looked at the corresponding Hamiltonian flow on M and showed that this flow descends to a Hamiltonian flow on the reduced space. For the uniqueness of the flow they also used the existence of a partition of unity, which is much easier to prove in their case using global action of G on (M^{2n}, ω).

5. A Leftschetz decomposition on a compact stratified symplectic space

In this section we show that a stratified symplectic space (X, ω) provided with a Poisson smooth structure $C^\infty(X)$ compatible with ω enjoys many properties related to the existence of a Leftschetz decomposition on (X, ω), see Lemma 5.1, Proposition 5.2, Theorem 5.4.

The notion of a Leftschetz decomposition on a symplectic manifold (M^{2n}, ω) has been introduced by Yan in [22], where he gives an alternative proof the Mathieu theorem on harmonic cohomology classes of (M^{2n}, ω) using the Leftschetz decomposition. His proof is considerably simpler than the earlier proof by Mathieu in [14]. Roughly speaking, a Leftschetz decomposition on a symplectic manifold (M^{2n}, ω) is an $\mathfrak{sl}_2$-module -structure of $\Omega(M^{2n})$. The Lie algebra $\mathfrak{sl}_2$ acting on $\Omega(M^{2n})$ is generated by linear operators L, L^*, A defined as follows. L is the wedge multiplication by ω, $L^* := i(G_\omega)$, and $A = [L^*, L]$. Now assume that (X^{2n}, ω) is a stratified symplectic space provided with a Poisson smooth structure $C^\infty(X^{2n})$, which is compatible with ω. By Proposition 3.11 $\Omega(X^{2n})$ is stable under L, L^*. Using the inclusion $\Omega_0(X^{reg}) \subset i^*(\Omega(X^{2n})) \subset \Omega(X^{reg})$, we get

Lemma 5.1. *The space $\Omega(X^{2n})$ is an $\mathfrak{sl}_2$-module, where $\mathfrak{sl}_2$ is the Lie algebra generated by $(L, L^*, A = [L, L^*])$.*

For any $k \geq 0$ set

$$\mathcal{P}_{n-k}(X^{2n}) := \{\alpha \in \Omega^{n-k}(X^{2n}) |\, L^{k+1}\omega = 0\}.$$

Elements of $\mathcal{P}_{n-k}(X^{2n})$ are called *primitive elements.*

Proposition 5.2. *We have the following decomposition for $k \geq 0$*

$$\Omega^{n-k}(X^{2n}) = \mathcal{P}_{n-k}(X^{2n}) \oplus L(\mathcal{P}_{n-k-2}(X^{2n})) \oplus \cdots \tag{5.1}$$

$$\Omega^{n+k}(X^{2n}) = L^k(\mathcal{P}_{n-k}(X^{2n})) \oplus L^{k+1}(\mathcal{P}_{n-k-2}(X^{2n})) \oplus \cdots \tag{5.2}$$

Proof. Using the analogous decomposition for $\Omega(X^{reg})$ [22, Corollary 2.6] we decompose each element $i^*(\alpha) \in i^*(\Omega^{n-k}(X^{2n}))$ as

$$i^*(\alpha) = \alpha_p^{n-k} + L(\alpha_p^{n-k-2}) + \cdots + L^{[(n-k)/2]}\alpha_p^{n-k-2[(n-k)/2]}, \tag{5.3}$$

and each $i^*(\beta) \in i^*(\Omega^{n+k}(X^{2n}))$ as

$$i^*(\beta) = L^k(\beta_p^{n-k}) + L^{k+1}(\beta_p^{n-k-2}) + \cdots , \tag{5.4}$$

where α_p^i, β_p^j are primitive elements of $\Omega^i(X^{reg})$ and $\Omega^j(X^{reg})$ respectively. To prove Proposition 5.2 it suffices to show that α_p^i, β_p^j are elements in $i^*(\Omega(X^{2n}))$. Now let us consider the decomposition of $i^*(\alpha) \in \Omega^{n-k}(X^{reg})$. We will show that all terms α_p^i can be obtained from $i^*(\alpha)$ inductively. First we assume that $n - k = 2q$, hence $\alpha_p^0 \in C^\infty(X^{reg})$. Applying to the both sides of (5.3) the operator L^{n-q} we get

$$L^{n-q}(i^*(\alpha)) = \omega^n \cdot \alpha_p^0 \in i^*(\Omega^{2n}(X^{2n})).$$

By Proposition 3.11, $\alpha_p^0 \in i^*(C^\infty(X^{2n}))$.

Now let us assume that $n - k = 2q + 1$. In the same way we have

$$L^{n-q-1}(i^*(\alpha)) = \omega^{n-1} \cdot \alpha_p^1 \in i^*(\Omega^{2n-1}(X^{2n})). \tag{5.5}$$

Lemma 5.3. *1. For any $\gamma \in \Omega^k(X^{reg})$ we have*

$$[L^r, L^*]\gamma = (r(k-n) + r(r-1))L^{r-1}.$$

2. If γ is a primitive element in $\Omega^{n-k}(X^{reg})$, then there exists a non-zero constant $c_{n,k}$ such that $\gamma = c_{n,k} \cdot (L^*)^k \circ (L^k\gamma)$.

Proof. The first assertion of Lemma 5.3 for $r = 1$ is well-known, see [22, Corollary 1.6]. For $r \geq 2$ we use the following formula

$$[L^r, L^*] = L[L^{r-1}, L^*] + [L, L^*]L^{r-1},$$

which leads to the first assertion of Lemma 5.3 by induction.

2. It is known that $\gamma \in \Omega^k(X^{reg})$ is primitive, if and only if $L^*(\gamma) = 0$ [22, Corollary 2.6]. Using this we get the second assertion of Lemma 5.3 by applying the first assertion recursively. □

Now let us continue the proof of Proposition 5.2. By Lemma 5.3 and by (5.5) the term α_p^1 can be obtained from $L^{n-p-1}(i^*(\alpha))$ by applying the operator $c_{n,1} \cdot (L^*)^{n-1}$. Clearly $\alpha_p^1 \in i^*(\Omega(X^{2n}))$.

Repeating this procedure, we get all terms α_p^i, which by Lemma 5.3 belongs to $i^*(\Omega(X^{2n}))$. In the same ways we prove that all terms β_p^j are elements of $i^*(\Omega(X^{2n}))$. This completes the proof of Proposition 5.2. □

Since $[L, d] = 0$ holds on $\Omega(X^{reg})$ and $i^*(\Omega(X^{2n}))$ is stable under the action of d and L, the equality $[L, d] = 0$ also holds on $\Omega(X^{2n})$. In particular, the wedge product with $[\omega^k]$ maps $H^{n-k}(\Omega(X^{2n}), d)$ to $H^{n+k}(\Omega(X^{2n}), d)$. A stratified symplectic space (X^{2n}, ω) equipped with a Poisson smooth structure $C^\infty(X^{2m})$ is said *to satisfy the hard Lefschetz condition*, if the cup product

$$[\omega^k] : H^{n-k}(\Omega(X^{2n}), d) \to H^{n+k}(\Omega(X^{2n}), d)$$

is surjective for any $k \le n = \frac{1}{2} \dim X^{2n}$. A differential form $\alpha \in \Omega(X^{2n})$ is called *harmonic*, if $d\alpha = 0 = \delta\alpha$. Let us abbreviate $H^*(\Omega(X^{2n}), d)$ by $H^*_{dR}(X^{2n})$.

Theorem 5.4. *Let (X^{2n}, ω) be a stratified symplectic space and $C^\infty(X^{2n})$ Poisson smooth structure which is also compatible with ω. Then the following two assertions are equivalent:*

1. *Any cohomology class in $H^*_{dR}(X^{2n})$ contains a harmonic cocycle.*
2. *(X^{2n}, ω) satisfies the hard Lefschetz condition.*

Proof. The proof of Theorem 5.4 for smooth symplectic manifold by Yan in [22, Theorem 0.1] can be repeated word-by-word. For the convenience of the reader we outline a proof here. Denote by $H^k_{hr}(X^{2n})$ the space of all harmonic k-forms on (X^{2n}, ω), and let $H^*_{hr} = \oplus_{i=0}^{2n} H^i_{hr}(X^{2n})$.

Now let us prove that the assertion (1) of Theorem 3.10 implies the assertion (2) of Theorem 3.10. We consider the following diagram

$$\begin{array}{ccc} H^{n-k}_{hr}(X^{2n}) & \xrightarrow{L^k} & H^{n+k}_{hr}(X^{2n}) \\ \downarrow & & \downarrow \\ H^{n-k}_{dR}(X^{2n}) & \xrightarrow{L^k} & H^{n+k}_{dR}(X^{2n}). \end{array}$$

Let us recall that $i : X^{reg} \to X^{2n}$ is the canonical inclusion. Since $[L, \delta] = -d$ [22, Lemma 1.2], which can be easily proved for (X^{2n}, ω) satisfying the condition of Theorem 5.4, Proposition 5.2 implies that $L^k : H^{n-k}_{hr}(X^{2n}) \to$

$H_{hr}^{n+k}(X^{2n})$ is an isomorphism. Since the vertical arrows in the diagram are surjective, we conclude that the second horizontal arrow in the diagram is also surjective. This proves (1) $\Longrightarrow$ (2).

Now let us prove that (2) $\Longrightarrow$ (1). Note that the condition (2) implies that [22, §3]

$$H_{dR}^{n-k}(X^{2n}) = Im\, L + P_{n-k},$$

where $P_{n-k} := \{\alpha \in H_{dR}^{n-k}(X^{2n}) |\, L^{k+1}\alpha = 0 \in H_{dR}^{n+k+2}(X^{2n})\}$.

Using induction argument, it suffices to prove that in each primitive cohomology class $v \in P_{n-k}$ there is a harmonic cocycle. Let $v = [z]$, $z \in \Omega^{n-k}(X^{2n})$. Since v is primitive we have $[z \wedge \omega^{k+1}] = 0 \in H_{dR}^{m+k+2}(X^{2n})$. Hence, $z \wedge \omega^{k+1} = d\gamma$ for some $\gamma \in \Omega^{n+k+1}(X^{2n})$. By Proposition 5.2 the operator $L^{k+1} : \Omega^{n-k-1}(X^{2n}) \to \Omega^{n+k+1}(X^{2n})$ is onto, consequently there exists $\theta \in \Omega^{m-k-1}(X^{2n})$ such that $\gamma = \theta \wedge \omega^{k+1}$. It follows that $(z - d\theta) \wedge \omega^{k+1} = 0$. Therefore $w = z - d\theta$ is primitive and closed. Since $[L^*, d] = \delta$ [22, Corollary 1.3], we get $\delta w = 0$. This completes the proof of Theorem 3.10.

□

Remark 5.5. 1. If $C^\infty(X^{2n})$ is locally smoothly contractible, by [17, Theorem 5.2] the de Rham cohomology $H^*(\Omega(X^{2n}), d)$ coincides with the singular cohomology $H^*(M, \mathbb{R})$), since X^{2n} admits smooth partitions of unity, see Proposition 2.17.

2. In [3, Proposition 5.4] Cavalcanti proved that the hard Lefschetz property on a compact symplectic manifold implies $Im\,\delta \cap \ker d = Im\, d \cap Im\,\delta$, see also [16]. His theorem can be proved word-by-word for stratified symplectic manifolds equipped with a Poisson smooth structure, since the main ingredient of the proof is Proposition 5.2.

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Hông Vân Lê, Institute of Mathematics of ASCR, Zitna 25, 11567 Praha 1, Czech Republic
Petr Somberg, Mathematical Institute, Charles University, Sokolovska 83, 180 00 Praha 8, Czech Republic.
Jiři Vanžura, Institute of Mathematics of ASCR, Zizkova 22, 61662 Brno, Czech Republic